

Gauge symmetries in geometric phases¹

Kazuo Fujikawa

*Institute of Quantum Science, College of Science and Technology
Nihon University, Chiyoda-ku, Tokyo 101-8308, Japan*

Abstract

The analysis of geometric phases is briefly reviewed by emphasizing various gauge symmetries involved. The analysis of geometric phases associated with level crossing is reduced to the familiar diagonalization of the Hamiltonian in the second quantized formulation. A hidden local gauge symmetry becomes explicit in this formulation and specifies physical observables; the choice of a basis set which specifies the coordinates in the functional space is arbitrary in the second quantization, and a sub-class of coordinate transformations, which keeps the form of the action invariant, is recognized as the gauge symmetry. It is shown that the hidden local symmetry provides a basic concept which replaces the notions of parallel transport and holonomy. We also point out that our hidden local gauge symmetry is quite different from a gauge symmetry used by Aharonov and Anandan in their definition of non-adiabatic phases.

1 Introduction

The conventional machinery to analyze geometric phases is a precise adiabatic approximation and the notions of parallel transport and holonomy[1]. The common less stringent but more flexible approaches are based on the practical Born-Oppenheimer approximation where the period T of the slower system in units of the time scale of the faster system is finite [2, 3, 4]. The idea of the non-adiabatic phase, which does not rely on the adiabatic approximation, has also been proposed [5, 6, 7]; the gauge symmetry used in this proposal is explained later.

Here I review a second quantized formulation of geometric phases[8, 9, 10] with an emphasis on gauge symmetries involved: The possible advantages of this formulation are

1. All the phases become purely dynamical, i.e., part of the Hamiltonian, and thus the analysis of geometric phases is reduced to a simple diagonalization of the Hamiltonian. This formulation also works for both of the path integral and operator formulations.
2. Origins of various gauge symmetries used in the past analyses become transparent. In particular, the presence of a hidden local gauge symmetry in the analysis of adiabatic phases is clearly recognized.
3. One obtains a better view of topological properties, such as the absence of the monopole-like singularity.

¹Talk given at Summer Institute 2005, Fuji-Yoshida, August 11-18, 2005 (to be published in Proceedings).

4. Differences between the quantum anomaly and the geometric phase become clear in this formulation.

From the point of view of particle physicists, the essence of the geometric phase may be summarized as follows: The essence of the geometric phase is to use an *approximation* (adiabatic approximation) to obtain a clear universal view of what is going on in the general level crossing problem, which is not clearly seen in the exact treatment. Mathematically, a time dependent unitary transformation which is singular at the level crossing point plays a central role.

2 Second quantized formulation and geometric phases

We start with the generic (hermitian) Hamiltonian

$$\hat{H} = \hat{H}(\hat{p}, \hat{x}, X(t)) \quad (1)$$

for a single particle theory in a slowly varying background variable $X(t) = (X_1(t), X_2(t), \dots)$. The path integral for this theory for the time interval $0 \leq t \leq T$ in the second quantized formulation is given by

$$\begin{aligned} & \int \mathcal{D}\psi^* \mathcal{D}\psi \exp\left\{\frac{i}{\hbar} \int_0^T dt d^3x [\mathcal{L}]\right\}, \\ \mathcal{L} = & \psi^*(t, \vec{x}) i\hbar \frac{\partial}{\partial t} \psi(t, \vec{x}) - \psi^*(t, \vec{x}) \hat{H}\left(\frac{\hbar}{i} \frac{\partial}{\partial \vec{x}}, \vec{x}, X(t)\right) \psi(t, \vec{x}) \end{aligned} \quad (2)$$

We then define a complete set of eigenfunctions

$$\begin{aligned} & \hat{H}\left(\frac{\hbar}{i} \frac{\partial}{\partial \vec{x}}, \vec{x}, X(0)\right) u_n(\vec{x}, X(0)) = \lambda_n u_n(\vec{x}, X(0)), \\ & \int d^3x u_n^*(\vec{x}, X(0)) u_m(\vec{x}, X(0)) = \delta_{nm}, \end{aligned} \quad (3)$$

and expand

$$\psi(t, \vec{x}) = \sum_n a_n(t) u_n(\vec{x}, X(0)). \quad (4)$$

We then have

$$\mathcal{D}\psi^* \mathcal{D}\psi = \prod_n \mathcal{D}a_n^* \mathcal{D}a_n \quad (5)$$

and the path integral is written as

$$\begin{aligned} Z &= \int \prod_n \mathcal{D}a_n^* \mathcal{D}a_n \exp\left\{\frac{i}{\hbar} \int_0^T dt \mathcal{L}\right\}, \\ \mathcal{L} &= \sum_n a_n^*(t) i\hbar \frac{\partial}{\partial t} a_n(t) - \sum_{n,m} a_n^*(t) E_{nm}(X(t)) a_m(t) \end{aligned} \quad (6)$$

where

$$E_{nm}(X(t)) = \int d^3x u_n^*(\vec{x}, X(0)) \hat{H}(\frac{\hbar}{i} \frac{\partial}{\partial \vec{x}}, \vec{x}, X(t)) u_m(\vec{x}, X(0)). \quad (7)$$

We next perform a unitary transformation

$$a_n(t) = \sum_m U(X(t))_{nm} b_m(t) \quad (8)$$

where

$$U(X(t))_{nm} = \int d^3x u_n^*(\vec{x}, X(0)) v_m(\vec{x}, X(t)) \quad (9)$$

with the instantaneous eigenfunctions of the Hamiltonian

$$\begin{aligned} \hat{H}(\frac{\hbar}{i} \frac{\partial}{\partial \vec{x}}, \vec{x}, X(t)) v_n(\vec{x}, X(t)) &= \mathcal{E}_n(X(t)) v_n(\vec{x}, X(t)), \\ \int d^3x v_n^*(\vec{x}, X(t)) v_m(\vec{x}, X(t)) &= \delta_{n,m}. \end{aligned} \quad (10)$$

We take the time T as a period of the slowly varying variable $X(t)$ in the analysis of geometric phases.

We can thus re-write the path integral as

$$\begin{aligned} Z &= \int \prod_n \mathcal{D}b_n^* \mathcal{D}b_n \exp\left\{\frac{i}{\hbar} \int_0^T dt \mathcal{L}\right\}, \\ \mathcal{L} &= \sum_n b_n^*(t) i\hbar \frac{\partial}{\partial t} b_n(t) - \sum_n b_n^*(t) \mathcal{E}_n(X(t)) b_n(t) \\ &\quad + \sum_{n,m} b_n^*(t) \langle n | i\hbar \frac{\partial}{\partial t} | m \rangle b_m(t) \end{aligned} \quad (11)$$

where the last term in the action stands for the term commonly referred to as Berry's phase and its off-diagonal generalization. This term is defined by

$$\begin{aligned} (U(X(t))^\dagger i\hbar \frac{\partial}{\partial t} U(X(t)))_{nm} &= \int d^3x v_n^*(\vec{x}, X(t)) i\hbar \frac{\partial}{\partial t} v_m(\vec{x}, X(t)) \\ &\equiv \langle n | i\hbar \frac{\partial}{\partial t} | m \rangle. \end{aligned} \quad (12)$$

The use of the instantaneous eigenfunctions here is a common feature shared with the adiabatic approximation. In our picture, all the information about geometric phases is included in the effective Hamiltonian and thus purely *dynamical*.

When one defines the Schrödinger picture by

$$\hat{\mathcal{H}}_{eff}(t) \equiv \sum_n \hat{b}_n^\dagger(0) \mathcal{E}_n(X(t)) \hat{b}_n(0) - \sum_{n,m} \hat{b}_n^\dagger(0) \langle n | i\hbar \frac{\partial}{\partial t} | m \rangle \hat{b}_m(0) \quad (13)$$

in the operator formulation, the second quantized formula for the evolution operator gives

$$\begin{aligned} & \langle m | T^* \exp \left\{ -\frac{i}{\hbar} \int_0^T \hat{\mathcal{H}}_{eff}(t) dt \right\} | n \rangle \\ &= \langle m(T) | T^* \exp \left\{ -\frac{i}{\hbar} \int_0^T \hat{H}(\hat{\vec{p}}, \hat{\vec{x}}, X(t)) dt \right\} | n(0) \rangle \end{aligned} \quad (14)$$

where T^* stands for the time ordering operation, and the state vectors in the second quantization on the left-hand side are defined by

$$|n\rangle = \hat{b}_n^\dagger(0)|0\rangle, \quad (15)$$

and the state vectors on the right-hand side stand for the first quantized states defined by

$$\langle \vec{x} | n(t) \rangle = v_n(\vec{x}, X(t)) \quad (16)$$

appearing in (10). Both-hand sides of the above equality (14) are exact, but the difference is that the geometric terms, both of diagonal and off-diagonal, are explicit in the second quantized formulation on the left-hand side.

The Schrödinger amplitude is given by

$$\psi_n(\vec{x}, T; X(T)) = \langle 0 | \hat{\psi}(T, \vec{x}) \hat{b}_n^\dagger(0) | 0 \rangle. \quad (17)$$

In the adiabatic approximation, we assume the dominance of diagonal elements

$$\begin{aligned} \psi_n(\vec{x}, T; X(T)) &= \sum_m v_m(\vec{x}; X(T)) \langle m | T^* \exp \left\{ -\frac{i}{\hbar} \int_0^T \hat{\mathcal{H}}_{eff}(t) dt \right\} | n \rangle \\ &\simeq v_n(\vec{x}; X(T)) \exp \left\{ -\frac{i}{\hbar} \int_0^T [\mathcal{E}_n(X(t)) - \langle n | i\hbar \frac{\partial}{\partial t} | n \rangle] dt \right\}. \end{aligned} \quad (18)$$

3 Hidden local gauge symmetry

The path integral formula (11) is based on the expansion

$$\psi(t, \vec{x}) = \sum_n b_n(t) v_n(\vec{x}, X(t)), \quad (19)$$

and the starting path integral (2) depends only on the field variable $\psi(t, \vec{x})$, not on $\{b_n(t)\}$ and $\{v_n(\vec{x}, X(t))\}$ separately. This fact shows that our formulation contains a hidden local gauge symmetry

$$\begin{aligned} v_n(\vec{x}, X(t)) &\rightarrow v'_n(t; \vec{x}, X(t)) = e^{i\alpha_n(t)} v_n(\vec{x}, X(t)), \\ b_n(t) &\rightarrow b'_n(t) = e^{-i\alpha_n(t)} b_n(t), \quad n = 1, 2, 3, \dots, \end{aligned} \quad (20)$$

where the gauge parameter $\alpha_n(t)$ is a general function of t . We call this symmetry "hidden local gauge symmetry" because it appears due to the separation of the fundamental dynamical variable $\psi(t, \vec{x})$ into two sets $\{b_n(t)\}$ and $\{v_n(\vec{x}, X(t))\}$. One can confirm that the action and the path integral measure in (11) are both invariant under this gauge transformation.

The above hidden local symmetry is exact as long as the basis set is not singular. In the present problem, the basis set becomes singular on top of level crossing, and thus the above symmetry is particularly useful in the general adiabatic approximation defined by the condition that the basis set is well-defined. Of course, one may consider a new hidden local gauge symmetry when one defines a new regular coordinate in the neighborhood of the singularity, and the freedom in the phase choice of the new basis set persists. Physically, this hidden gauge symmetry arises from the fact that the choice of the basis set which specifies the coordinates in the functional space is arbitrary in field theory, as long as the coordinates are not singular.

In practical applications for generic eigenvalues $\{\mathcal{E}_n(X(t))\}$, the sub-group

$$U = U(1) \times U(1) \times \dots \quad (21)$$

as in the above (20) is useful, because it keeps the form of the action invariant and thus becomes a symmetry of quantized theory in the conventional sense. In particular, it is exactly preserved in the adiabatic approximation in which the mixing of different energy eigenstates is assumed to be negligible and thus the coordinates specified are always well-defined.

For a special case where the first eigenvalue $\mathcal{E}_1(X(t))$ has n_1 -fold degeneracy, the second eigenvalue $\mathcal{E}_2(X(t))$ has n_2 -fold degeneracy, and so on, the sub-group

$$U = U(n_1) \times U(n_2) \times \dots, \quad (22)$$

which keeps the form of the action invariant, will be useful.

The above hidden local gauge symmetry is an exact symmetry of quantum theory, and thus physical observables in the adiabatic approximation should respect this symmetry. Also, by using this local gauge freedom, one can choose the phase convention of the basis set $\{v_n(t, \vec{x}, X(t))\}$ such that the analysis of geometric phases becomes most transparent.

Under the hidden local symmetry, the probability amplitude in (17) transforms

$$\psi'_n(\vec{x}, t; X(t)) = e^{i\alpha_n(0)} \psi_n(\vec{x}, t; X(t)) \quad (23)$$

independently of the value of t . Thus the product

$$\psi_n(\vec{x}, 0; X(0))^* \psi_n(\vec{x}, T; X(T)) \quad (24)$$

defines a manifestly gauge invariant quantity, namely, it is independent of the choice of the phase convention of the complete basis set $\{v_n(t, \vec{x}, X(t))\}$.

For the adiabatic formula (18), the gauge invariant quantity is given by

$$\begin{aligned} \psi_n(\vec{x}, 0; X(0))^* \psi_n(\vec{x}, T; X(T)) &= v_n(0, \vec{x}; X(0))^* v_n(T, \vec{x}; X(T)) \\ &\times \exp\left\{-\frac{i}{\hbar} \int_0^T [\mathcal{E}_n(X(t)) - \langle n | i\hbar \frac{\partial}{\partial t} | n \rangle] dt\right\} \end{aligned} \quad (25)$$

and the combination

$$v_n(0, \vec{x}; X(0))^* v_n(T, \vec{x}; X(T)) \exp\left\{-\frac{i}{\hbar} \int_0^T [-\langle n | i\hbar \frac{\partial}{\partial t} | n \rangle] dt\right\} \quad (26)$$

is invariant under the hidden local gauge symmetry. By choosing the gauge such that

$$v_n(T, \vec{x}; X(T)) = v_n(0, \vec{x}; X(0)) \quad (27)$$

the prefactor $v_n(0, \vec{x}; X(0))^* v_n(T, \vec{x}; X(T))$ becomes real and positive. Note that we are assuming the cyclic motion of the external parameter, $X(T) = X(0)$. Then the factor

$$\exp\left\{-\frac{i}{\hbar} \int_0^T [\mathcal{E}_n(X(t)) - \langle n | i\hbar \frac{\partial}{\partial t} | n \rangle] dt\right\} \quad (28)$$

extracts all the information about the phase in (25) and defines a physical quantity. After this gauge fixing, the above quantity is still invariant under residual gauge transformations satisfying the periodic boundary condition

$$\alpha_n(0) = \alpha_n(T), \quad (29)$$

in particular, for a class of gauge transformations defined by $\alpha_n(X(t))$. Note that our gauge transformation, which is defined by an arbitrary function $\alpha_n(t)$, is more general. It has been shown that this hidden gauge symmetry replaces the notions of parallel transport and holonomy in the analysis of geometric phases. We note that the notion such as holonomy is valid only in the limit of the very precise adiabatic approximation [1].

4 Explicit example; two-level truncation

We analyze the crossing of two levels in the above general model. In the sufficiently close to the level crossing point, we assume that we can truncate the model to a simplified two-level model which contains the two levels at issue. The effective Hamiltonian for the Lagrangian in (6) is then reduced to the 2×2 matrix $h(X(t)) = (E_{nm}(X(t)))$. If one assumes that the level crossing takes place at the origin of the parameter space $X(t) = 0$, one analyzes the matrix

$$h(X(t)) = (E_{nm}(0)) + \left(\frac{\partial}{\partial X_k} E_{nm}(X) |_{X=0} \right) X_k(t) \quad (30)$$

for sufficiently small $(X_1(1), X_2(1), \dots)$. After a suitable definition of the parameters $y(t)$ by taking linear combinations of $X_k(t)$, we write the matrix

$$h(X(t)) = \begin{pmatrix} E(0) + y_0(t) & 0 \\ 0 & E(0) + y_0(t) \end{pmatrix} + g \sigma^l y_l(t) \quad (31)$$

where σ^l stands for the Pauli matrices, and g is a suitable (positive) coupling constant.

The above matrix is diagonalized in the standard way as

$$h(X(t))v_{\pm}(y) = (E(0) + y_0(t) \pm gr)v_{\pm}(y) \quad (32)$$

where $r = \sqrt{y_1^2 + y_2^2 + y_3^2}$ and

$$\begin{aligned} v_+(y) &= \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\varphi} \\ \sin \frac{\theta}{2} \end{pmatrix}, \\ v_-(y) &= \begin{pmatrix} \sin \frac{\theta}{2} e^{-i\varphi} \\ -\cos \frac{\theta}{2} \end{pmatrix} \end{aligned} \quad (33)$$

by using the polar coordinates, $y_1 = r \sin \theta \cos \varphi$, $y_2 = r \sin \theta \sin \varphi$, $y_3 = r \cos \theta$. If one defines

$$v_m^\dagger(y) i \frac{\partial}{\partial t} v_n(y) = A_{mn}^k(y) \dot{y}_k \quad (34)$$

where m and n run over \pm , we have

$$\begin{aligned} A_{++}^k(y) \dot{y}_k &= \frac{(1 + \cos \theta)}{2} \dot{\varphi} \\ A_{+-}^k(y) \dot{y}_k &= \frac{\sin \theta}{2} \dot{\varphi} + \frac{i}{2} \dot{\theta} = (A_{-+}^k(y) \dot{y}_k)^*, \\ A_{--}^k(y) \dot{y}_k &= \frac{(1 - \cos \theta)}{2} \dot{\varphi}. \end{aligned} \quad (35)$$

The effective Hamiltonian corresponding to the Lagrangian (11) is then given by

$$\begin{aligned} \hat{H}_{eff}(t) &= (E(0) + y_0(t) + gr(t)) \hat{b}_+^\dagger \hat{b}_+ + (E(0) + y_0(t) - gr(t)) \hat{b}_-^\dagger \hat{b}_- \\ &\quad - \hbar \sum_{m,n} \hat{b}_m^\dagger A_{mn}^k(y) \dot{y}_k \hat{b}_n \end{aligned} \quad (36)$$

which is *exact* in the present two-level truncation.

In the conventional adiabatic approximation, one approximates the effective Hamiltonian by

$$\begin{aligned} \hat{H}_{eff}(t) &\simeq (E(0) + y_0(t) + gr(t)) \hat{b}_+^\dagger \hat{b}_+ + (E(0) + y_0(t) - gr(t)) \hat{b}_-^\dagger \hat{b}_- \\ &\quad - \hbar [\hat{b}_+^\dagger A_{++}^k(y) \dot{y}_k \hat{b}_+ + \hat{b}_-^\dagger A_{--}^k(y) \dot{y}_k \hat{b}_-] \end{aligned} \quad (37)$$

which is valid for

$$Tgr(t) \gg \hbar\pi, \quad (38)$$

where $\hbar\pi$ stands for the magnitude of the geometric term times T . The amplitude $\langle 0 | \hat{\psi}(T) \hat{b}_-^\dagger(0) | 0 \rangle$, which corresponds to the probability amplitude in the first quantization, is given by

$$\begin{aligned} \psi_-(T) &\equiv \langle 0 | \hat{\psi}(T) \hat{b}_-^\dagger(0) | 0 \rangle \\ &= \exp\left\{-\frac{i}{\hbar} \int_0^T dt [E(0) + y_0(t) - gr(t) - \hbar A_{--}^k(y) \dot{y}_k]\right\} v_-(y(T)) \end{aligned} \quad (39)$$

For a 2π rotation in φ with fixed θ , for example, the gauge invariant quantity gives rise to

$$\begin{aligned}
\psi_{-}(0)^{\star}\psi_{-}(T) &= v_{-}(y(0))^{\star}v_{-}(y(T)) \\
&\times \exp\left\{-\frac{i}{\hbar}\int_0^T dt[E(0) + y_0(t) - gr(t) - \hbar A_{-}^k(y)\dot{y}_k]\right\} \\
&= \exp\{i\pi(1 - \cos\theta)\} \\
&\times \exp\left\{-\frac{i}{\hbar}\int_{C_1(0\rightarrow T)} dt[E(0) + y_0(t) - gr(t)]\right\}
\end{aligned} \tag{40}$$

by using $v_{-}(y(T)) = v_{-}(y(0))$ in the present choice of gauge, and the path $C_1(0 \rightarrow T)$ specifies the integration along the above specific closed path.

The first phase factor $\exp\{i\pi(1 - \cos\theta)\}$ stands for the familiar Berry's phase and the second phase factor stands for the conventional dynamical phase. The phase factor is still invariant under a smaller set of gauge transformations with

$$\alpha_{-}(T) = \alpha_{-}(0) \tag{41}$$

and, in particular, for the gauge parameter of the form $\alpha_{-}(y(t))$.

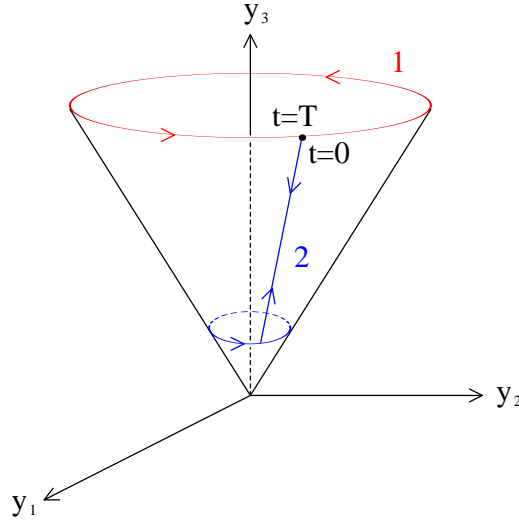


Fig. 1: The path 1 gives the conventional geometric phase for a fixed finite T , whereas the path 2 gives a trivial geometric phase for a fixed finite T . Note that both of the paths cover the same solid angle $2\pi(1 - \cos\theta)$.

By deforming the path 1 to the path 2 in the parameter space in Fig. 1, it is shown that the amplitude (40) is replaced by [8, 9]

$$\psi_{-}(0)^{\star}\psi_{-}(T) = \exp\left\{-\frac{i}{\hbar}\int_{C_2(0\rightarrow T)} dt[E(0) + y_0(t) - gr(t)]\right\} \tag{42}$$

The path $C_2(0 \rightarrow T)$ specifies the path 2 in Fig.1, and $v_-(y(T)) = v_-(y(0))$ in the present choice of the gauge. Thus no geometric phase for the path C_2 for any fixed finite T .

For $t = 0$ or $t = T$, we start or end with the parameter region where the condition for the adiabatic approximation is satisfied. But approaching the infinitesimal neighborhood of the origin where the level crossing takes place, the condition (38) is no more satisfied and instead one has $Tgr \ll \hbar$. In this region of the parameter space, \hat{H}_{eff} is replaced by

$$\hat{H}_{eff}(t) \simeq (E(0) + y_0(t))\hat{c}_+^\dagger\hat{c}_+ + (E(0) + y_0(t))\hat{c}_-^\dagger\hat{c}_- - \hbar\dot{\phi}\hat{c}_+^\dagger\hat{c}_+ \quad (43)$$

by performing a unitary transformation to a regular basis

$$\hat{b}_m = \sum_n U(\theta(t))_{mn}\hat{c}_n \quad (44)$$

with

$$U(\theta(t)) = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \quad (45)$$

We thus conclude that the topological interpretation of the Berry's phase fails in the practical Born-Oppenheimer approximation where T is identified with the period of the slower dynamical system.

5 Gauge symmetry in non-adiabatic phase

We briefly comment on the gauge symmetry involved in the non-adiabatic phase proposed by Aharonov and Anandan [5]. The analysis starts with the wave function satisfying

$$\int d^3x \psi(t, \vec{x})^* \psi(t, \vec{x}) = 1, \quad (46)$$

and a cyclic condition

$$\psi(T, \vec{x}) = e^{i\phi} \psi(0, \vec{x}) \quad (47)$$

with a real constant ϕ . These properties then imply the existence of a hermitian Hamiltonian

$$i\hbar \frac{\partial}{\partial t} \psi(t, \vec{x}) = \hat{H}(t, \frac{\hbar}{i} \frac{\partial}{\partial \vec{x}}, \vec{x}) \psi(t, \vec{x}) \quad (48)$$

but the motion of $X(t)$ is arbitrary.

The analysis of non-adiabatic phases is based on the equivalence class which identifies all the vectors of the form

$$\{e^{i\alpha(t)} \psi(t, \vec{x})\}. \quad (49)$$

The conventional Schrödinger equation is not invariant under this equivalence class, we may thus define an equivalence class of Hamiltonians

$$\{\hat{H} - \hbar \frac{\partial}{\partial t} \alpha(t)\}. \quad (50)$$

We next define an object [6, 7]

$$\Psi(t, \vec{x}) \equiv \exp[i \int_0^t dt \int d^3x \psi(t, \vec{x})^* i \frac{\partial}{\partial t} \psi(t, \vec{x}) \psi(t, \vec{x})] \quad (51)$$

which satisfies

$$\begin{aligned} \Psi(0, \vec{x}) &= \psi(0, \vec{x}), \\ \int d^3x \Psi(t, \vec{x})^* i \frac{\partial}{\partial t} \Psi(t, \vec{x}) &= 0. \end{aligned} \quad (52)$$

Under the equivalence class transformation (or gauge transformation)

$$\psi(t, \vec{x}) \rightarrow e^{i\alpha(t)} \psi(t, \vec{x}), \quad (53)$$

$\Psi(t, \vec{x})$ transforms as

$$\Psi(t, \vec{x}) \rightarrow e^{\alpha(0)} \Psi(t, \vec{x}). \quad (54)$$

The gauge invariant quantity is then defined by

$$\Psi(0, \vec{x})^* \Psi(T, \vec{x}) = \psi(0, \vec{x})^* \exp[i \int_0^T dt \int d^3x \psi(t, \vec{x})^* i \frac{\partial}{\partial t} \psi(t, \vec{x})] \psi(T, \vec{x}) \quad (55)$$

By a suitable gauge transformation

$$\psi(t, \vec{x}) \rightarrow \tilde{\psi}(t, \vec{x}) = e^{-i\alpha(t)} \psi(t, \vec{x}) \quad (56)$$

with

$$\alpha(T) - \alpha(0) = \phi \quad (57)$$

we can make the prefactor

$$\begin{aligned} \psi(0, \vec{x})^* \psi(T, \vec{x}) \rightarrow \tilde{\psi}(0, \vec{x})^* \tilde{\psi}(T, \vec{x}) &= e^{i\alpha(0)} \psi(0, \vec{x})^* e^{-i\alpha(T)} \psi(T, \vec{x}) \\ &= |\psi(0, \vec{x})|^2 \end{aligned} \quad (58)$$

real and positive for a cyclic evolution. The above gauge invariant quantity is then given by

$$\Psi(0, \vec{x})^* \Psi(T, \vec{x}) = |\psi(0, \vec{x})|^2 \exp[i \int_0^T dt \int d^3x \tilde{\psi}(t, \vec{x})^* i \frac{\partial}{\partial t} \tilde{\psi}(t, \vec{x})] \quad (59)$$

and the factor on the exponential extracts all the information about the phase from the gauge invariant quantity.

The "non-adiabtic" phase is then defined by

$$\beta = \oint dt \int d^3x \tilde{\psi}(t, \vec{x})^* i \frac{\partial}{\partial t} \tilde{\psi}(t, \vec{x}) \quad (60)$$

with

$$\tilde{\psi}(0, \vec{x}) = \tilde{\psi}(T, \vec{x}) \quad (61)$$

This phase describes certain intrinsic properties of the Schrödinger equation, and it is invariant under a residual gauge symmetry with

$$\alpha(T) = \alpha(0). \quad (62)$$

We emphasize that the gauge symmetry here is quite different from our hidden local symmetry which is related to an arbitrariness of the choice of coordinates in the functional space.

The geometric phases in classical mechanics have also been analyzed in the literature [11].

References

- [1] B. Simon, Phys. Rev. Lett. **51**, 2167 (1983).
- [2] M.V. Berry, Proc. Roy. Soc. Ser. A**392**, 45 (1984).
- [3] A.J. Stone, Proc. Roy. Soc. Ser. A**351**, 141 (1976).
- [4] M.V. Berry, Proc. Roy. Soc. Ser. A**414**, 31 (1987).
- [5] Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987).
- [6] I.J.R. Aitchison and K. Wanelik, Proc. Roy. Soc. Ser. A**439**, 25 (1992).
- [7] N. Mukunda and R. Simon, Ann. Phys. (N.Y.) **228**, 205 (1993).
- [8] K. Fujikawa, Mod. Phys. Lett. A**20**, 335 (2005).
- [9] S. Deguchi and K. Fujikawa, Phys. Rev. A**72**, 012111 (2005).
- [10] K. Fujikawa, Phys. Rev. D**72**, 025009 (2005).
- [11] For a recent account of geometric phases see, for example, D. Chruscinski and A. Jamiolkowski, *Geometric Phases in Classical and Quantum Mechanics* (Birkhauser, Berlin, 2004).